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IMMERSIONS OF POSITIVELY CURVED MANIFOLDS INTO MANIFOLDS WITH CURVATURE BOUNDED ABOVE

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ABSTRACT. Let M be a compact, connected, orientable Riemannian manifold of dimension $n-1 \geq 2$, and let x be an isometric immersion of M into an n-dimensional Riemannian manifold N. Let K denote sectional curvature and i denote the injectivity radius. Assume, for some constant positive constant c, that $K(N) \leq 1/(4c^2)$, $1/c^2 \leq K(M)$, and $\pi c \leq i(N)$. Then the radius of the smallest N-ball containing $x(M)$ is less than $\frac{1}{2}\pi c$ and x is in fact an imbedding of M into N, whose image bounds a convex body.

1. INTRODUCTION

We shall investigate the following questions. If a compact manifold M_1 with positive sectional curvature, is isometrically immersed in some ambient space N, what is the radius of the smallest ball in which its image lies? Additionally, when can the knowledge that the image lies inside a ball of restricted size be used to conclude that the immersion is an imbedding whose image bounds a convex body in N?

The question of when an immersion is in fact an imbedding whose image bounds a convex body was first considered by Hadamard. The most complete theory concerns immersions of nonnegatively curved hypersurfaces in Euclidean space. In this case, work of Hadamard [7], Stoker [13], and Chern and Lashof [6] culminated in the theorem of Sacksteder [10] and van Heijenoort [8]. According to this theorem, if M is a complete, connected Riemannian manifold of dimension $n-1$ (throughout this paper we always assume that $n-1 \geq 2$) with $K(M) \geq 0$ and with at least one sectional curvature positive, and if $x: M \to R^n$ is an isometric immersion, then x is an imbedding and $x(M)$ bounds a convex body. Do Carmo and Lima [4] and Wu [17] have developed alternative approaches to part of this material, and have obtained additional information about the images of the immersion.

Ambient spaces other than Euclidean space have also been considered. Do Carmo and Warner [5] showed that if the ambient space is a complete, simply connected space of constant curvature, K, and M is complete and connected with $K(M) \geq K$, then any immersion $x: M \to N$ is an imbedding and $x(M)$ bounds a convex body.

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We know of two theorems concerning the case where N has variable curvature. Alexander has considered the case where the ambient manifold, N, is a Hadamard manifold; that is, where N is complete and simply connected with nonpositive curvature [1]. Let x be a hypersurface immersion of a compact, connected, orientable manifold M of dimension $n-1$ into N, and assume the existence of a continuous unit normal field along x with respect to which x has a positive semidefinite second fundamental form. Again the conclusion is that x imbeds M into N as the boundary of a convex body. Here the hypotheses concern the second fundamental form of the immersion rather than the sectional curvature of M; however, if $K(M) > 0$ then the hypotheses are satisfied.

The case where N has variable positive curvature was considered by Tribuzy [15]. There it was assumed that N is complete, orientable, simply connected, noncompact, and satisfies $K \geq K(N) > 0$ for some positive constant K. Let x be a hypersurface immersion of a compact, connected, orientable manifold M of dimension $n-1$ into N, and assume the existence of a continuous unit normal field along x with respect to which x has a second fundamental form whose eigenvalues are all at least $2\sqrt{K_0}$. Then x imbeds M into N as the boundary of a convex body. This hypothesis on the eigenvalues of the second fundamental form implies that $K(M) > 4K_0$.

The following theorem will be proved in §3.2.

Theorem 2. *Let M be a compact, connected, orientable Riemannian manifold of dimension $n-1$, with $n-1 \geq 2$, and $K(M) \geq 1/c^2$, where c is a positive constant. Let N be an n-dimensional Riemannian manifold such that $\pi c \leq i(N)$ and $K(N) \leq 1/(4c^2)$. Then any isometric immersion $x\colon M \to N$ imbeds M into N as the boundary of a convex body.*

In the above theorem $i(N)$ is the injectivity radius of N. This theorem includes the theorems of Hadamard, Tribuzy, and Alexander in the case $K(M) > 0$. If N is a simply connected space of constant curvature K, then the theorem of do Carmo and Warner is stronger than ours, in that they only require $K(M) \geq K$ whereas we have $K(M) \geq 4K$. Note that in Theorem 2 the curvature of N is allowed to be both positive and negative, whereas in all previous cases $K(N)$ was restricted to be either positive or nonpositive. Furthermore, rather than assuming N to be simply connected, we have placed a certain lower bound on the injectivity radius $i(N)$ of N. For example, real projective spaces of curvature less than or equal to $1/(4c^2)$ satisfy the conditions of Theorem 2.

The proof of Theorem 2 relies on the fact that $x(M)$ may be shown to lie in a ball with radius less than $\frac{1}{2}\pi c$. Spruck [12] showed that the smallest Euclidean ball in R^n containing a compact, connected Riemannian manifold M of dimension $n-1$ with $K(M) \geq 1/c^2$ has radius $r < \frac{1}{2}\pi c$ and this bound is the best possible. The following theorem is a generalization of Spruck's result.

Theorem 1. *Let M be a compact, connected Riemannian manifold of dimension*

m, $m \geq 2$, with $K(M) \geq 1/c^2$, where c is a positive constant. Let N be an n-dimensional Riemannian manifold such that $\pi c \leq i(N)$ and $K(N) \leq 1/(4c^2)$. If $x\colon M \to N$ is an isometric immersion, then $x(M)$ is contained in a metric ball of N with radius $R < \frac{1}{2}\pi c$.

The bound in Theorem 1 is the best possible, in that for any $\varepsilon > 0$ and simply connected space N of constant curvature less than or equal to $1/(4c^2)$, there exist an M satisfying the above conditions and an immersion $x\colon M \to N$ such that $x(M)$ lies in no ball of radius $\frac{1}{2}\pi c - \varepsilon$.

The fact that the bound on R does not change with the curvature of N may seem surprising at first, since the extrinsic radius of an immersed sphere of intrinsic curvature $1/c^2$ does change as the curvature of the ambient space changes.

The conditions on M in Theorem 1 are less restrictive than those of Theorem 2, where M is both orientable and of codimension 1. In Theorem 1, M need not be orientable and can be of any codimension.

This paper is based on the author's doctoral thesis presented at the University of Illinois. The author thanks her advisor, Professor Stephanie Alexander, for the valuable criticism she has given during the preparation of the thesis.

2. The radius of the smallest ball containing a compact manifold of positive curvature

2.1. Background material. All manifolds are assumed to be C^∞. We use the term immersion to mean immersion. All balls are assumed to be closed unless otherwise described. The convention that $1/d^2 = 0$ when $d = \infty$ will be used throughout. Thus, the phrase "a sphere of curvature $1/d^2$" includes the Euclidean case unless it is specifically excluded.

A *convex set* X in a complete Riemannian manifold will be one in which every two points of X are joined by a minimizing geodesic which lies in X and is unique in X. The closed metric ball $B(x, r)$ of radius r at x is called *strongly convex* if every two points, p and q, in $B(x, r)$ are joined by a unique minimizing geodesic, pq, and the interior of pq (the open arc) lies in the interior of $B(x, r)$. We assume pq is parametrized by arclength and let $\overline{pq}$ represent the vector tangent to pq with length equal to the length of pq. The *radius of convexity* at p is the largest r_0 such that for all $r < r_0$ the closed ball $B(p, r)$ is strongly convex.

Bonnet's result that the diameter, $d(M)$, of a compact manifold M of curvature at least $1/c^2$ satisfies $d(M) \leq \pi c$ will be referred to as Bonnet's Theorem [3, p. 27]. Similarly, Toponogov's result that the length of a geodesic triangle on a manifold of curvature at least $1/c^2$ does not exceed $2\pi c$ will be referred to as Toponogov's Theorem [14, p. 326]. A *geodesic triangle* is a union of three minimizing geodesics.

2.2. The smallest ball containing a compact set. If X is a compact subset of a complete Riemannian manifold, it is easy to see that there exists a closed ball of smallest radius r containing X. In general there may not be a unique smallest ball; consider the equator of a sphere. On a right cylinder, a latitudinal circle is contained in an infinite number of smallest balls, one centered at each point on the circle. In the last example the smallest balls are not normal coordinate balls.

Let N be a Riemannian manifold and $B(p, r)$ be a closed normal coordinate ball at p. We say a set $X \subset B(p, r)$ *supports* $B(p, r)$ if for each v in N_p the exponentiated hemisphere $\{\exp_p(u): u \cdot v \geq 0, |u| = r\}$ contains a point of X.

Lemma 1. *Let N be a complete Riemannian manifold and X a compact subset of N. Let $B(p, R)$ be a closed ball of smallest radius containing X. If $B(p, R)$ is a normal coordinate ball then X supports $B(p, R)$.*

Proof. Suppose that X does not support $B = B(p, R)$. Let $v \in N_p$ be a direction for which there is no support; i.e., suppose the set $\{\exp_p(u): u \cdot v \geq 0, |u| = R\}$ contains no points of X. Since X and ∂B are compact, $X \cap \partial B$ is compact and there is a smallest closed and truncated solid cone C in N_p centered about $-v$ such that $\exp_p C$ contains $X \cap \partial B$. Thus, C is the set of vectors $\{u \in N_p: |u| \leq R \text{ and } u \cdot v \leq a\}$ for some $a < 0$. Let γ be the geodesic starting at p with direction $-v$. It is straightforward using the first variation equation to show that $\gamma(s)$, for small s, is the center of a ball containing X and having radius r less than R.

The ball $B = B(p, R)$ in the statement of Lemma 1 is guaranteed to be unique under certain conditions. In the following proposition, the assumption that R be strictly less than the radius of convexity cannot be relaxed, as the example of the equator of a sphere shows.

Proposition 1. *Let N be a complete Riemannian manifold and X a compact subset of N. Let $B(p, R)$ be a closed ball of smallest radius containing X. If $B(p, R)$ is a normal coordinate ball and R is less than the radius of convexity at every x in X, then $B(p, R)$ is the unique ball of smallest radius containing X.*

Proof. Suppose $B(q, R)$ also contains X, where $q \neq p$. Let γ be a geodesic ray from p through q and pq the correpsonding segment. Let x_0 be the point of X on the exponentiated hemisphere associated with $-\gamma$. Then $d(p, x_0) = R$ and $d(q, x_0) \leq R$. Thus pq lies in the strongly convex ball $B(x_0, R)$. Note that $B(x_0, R)$ lies in a larger strongly convex ball and thus in an open normal coordinate ball. Now for $-\delta < s < \delta$, there is a smooth variation of geodesics $\alpha(s, t)$ for which $\alpha(s_0, t)$ is a unit speed geodesic from $\gamma(s_0)$ to x_0. Then at p

$$\frac{dL(\alpha)}{ds} = -|\overline{px}_0|^{-1}(\langle \overline{pq}, \overline{px}_0 \rangle).$$

Therefore, the angle between pq and px_0 is not obtuse since if it were, then

$dL(\alpha)/ds$ would be positive at p and $|\overline{\gamma(s)x_0}|$ would exceed R for small positive s.

By construction the angle x_0pq is not acute and hence it is a right angle. This is impossible since the interior of pq lies in the interior of $B(x_0, R)$. Indeed, if the angle x_0pq were a right angle there would be a geodesic segment pz near pq, lying in the interior of $B(x_0, R)$ except for p, such that the angle x_0pz is obtuse. As we have already seen, by the first variation formula, no such segment exists, and so the proof is complete.

2.3. Triangles on a sphere. We shall use the following corollary to the Rauch Comparison Theorem. A proof may be found in [3, p. 30].

Proposition 2. *Let N be a Riemannian manifold of dimension n $(n > 2)$ and $K(N) \leq 1/d^2$. Let S^n be an n-sphere of curvature $1/d^2$. Assume that $I\colon N_p \to S^n_q$ is a linear injection preserving inner products. If $r < \min\{i(N), \pi d\}$, then for any curve α in the closed ball $B(p, r)$ in N, we have*

$$L(\alpha) \geq L(\exp_q \circ I \circ (\exp_p)^{-1}(\alpha)).$$

Some of our comparisons will be based on the following fact from spherical trigonometry.

Lemma 2. *Let S^n be a sphere of dimension n $(n \geq 2)$ and constant curvature $1/d^2$, $d < \infty$. Let p_1 and p_2 be points of distance R from a third point p, where $r \leq \frac{1}{2}\pi d$. Then*

$$d(p_1, p_2) = 2d \cdot \sin^{-1}(\sin(R/d)\sin(\tfrac{1}{2}\theta))$$

where θ is the measure of the angle between pp_1 and pp_2.

This formula corresponds to the formula

$$d(p_1, p_2) = 2R\sin(\tfrac{1}{2}\theta)$$

in the Euclidean case.

Let $B(p, R)$ be a smallest ball in N containing the compact set X. We shall need a lower bound for the maximum length of a geodesic triangle formed with vertices in $X \cap \partial B(p, R)$. In the Euclidean case, the bound was computed by Spruck [12]. Lemma 3, which follows, gives the bound in the case where N is a sphere of constant curvature $1/d^2$.

Lemma 3. *Let N be a sphere of constant curvature $1/d^2$ and X be a compact set in N. Let $B(p, R)$ be a ball of smallest radius in N containing X. Then if $R \leq \frac{1}{2}\pi d$, there is a geodesic triangle whose vertices are points of X and whose length is at least the minimum of $4R$ and $6d \cdot \sin((1/\sqrt{2})\sin(R/d))$. If $d = \infty$, then there exists a geodesic triangle whose vertices are points of X and whose length is at least $4R$.*

Proof. Let $B = B(p, R)$. Suppose $d < \infty$. By Lemma 1, we know that X supports B. Choose p_1 and p_2 in $X \cap \partial B$ so that $d(p_1, p_2)$ equals the maximal distance between any two points of $X \cap \partial B$. Let θ be the measure of angle p_1pp_2. Because X supports B we have $\frac{1}{2}\pi < \theta \leq \pi$.

If $\frac{1}{2}\pi \leq \theta < \pi$, let v be the unit bisector in N_p of the angle determined by $\overline{pp}_1$ and $\overline{pp}_2$. Let

$$Y = \{w \in N_p : |w| = R \text{ and } w \cdot v \leq 0\}.$$

The exponentiated hemisphere, $\exp_p Y$, must contain at least one point p_3 of $X \cap \partial B$. Let $p_4 = \exp_p(Rv)$. If $(\overline{pp}_3, v) < 0$, then there is a point p_5 of ∂B such that $(\overline{pp}_5, v) = 0$ and pp_5 is interior to the angle of p_3pp_4. Let $p_5 = p_3$ if $\langle \overline{pp}_3, v\rangle = 0$. Thus we have $L(p_1p_2p_5) \leq L(p_1p_2p_3)$ where $L(p_ip_jp_k)$ is the length of triangle $p_ip_jp_k$. We shall minimize the length $L = L(p_1p_2p_5)$.

Let β be the measure of the angle p_1pp_5. By symmetry the measure of the angle formed by pp_2 and the ray opposite pp_5 is also β. Thus the measure of the angle p_2pp_5 is $\pi - \beta$. From Lemma 2 we have

$$L = 2d[\sin^{-1}(\sin(R/d) \cdot \sin(\tfrac{1}{2}\theta)) + \sin^{-1}(\sin(R/d) \cdot \sin(\tfrac{1}{2}\beta)) \\ + \sin^{-1}(\sin(R/d) \cdot \cos(\tfrac{1}{2}\beta))].$$

By the choice of p_1 and p_2, both β and $\pi - \beta$ are less than θ and hence $\pi - \theta \leq \beta \leq \theta$. For a fixed θ, L is minimized at an extreme value for β. Indeed, L is a concave function of θ and symmetric in β about $\frac{1}{2}\pi$.

At an extreme value of β,

$$L = 2d[2\sin^{-1}(\sin(R/d) \cdot \sin(\tfrac{1}{2}\theta)) + \sin^{-1}(\sin(R/d) \cdot \cos(\tfrac{1}{2}\theta))].$$

This expression of L is minimized at one of the extreme values of θ since L is a concave function of θ. Therefore L is minimized as θ approaches $\frac{1}{2}\pi$ or π. In the first case the expression becomes

$$L = 6d[\sin^{-1}((1/\sqrt{2})\sin(R/d))]$$

and in the second, $L = 4R$.

2.4. Proof of Theorem 1. By Bonnet's theorem, any two points of M can be joined by an M-geodesic of length less than or equal to πc. By a theorem of Synge [2, p. 194], in order that $x \circ \gamma_s$ be a geodesic of N, for a curve γ of M, we must have $K(M)(P) \leq K(N)(X_*(P))$ for any plane section P of M tangent to γ. Since $K(N)$ is known to be less than $K(M)$ for all plane sections P and since M is complete, we can conclude that $d(q_1, q_2)$ strictly exceeds $d(x(q_1), x(q_2))$ for any q_1 and q_2 in M. It follows that any two points of $x(M)$ can be joined by an N-geodesic of length less than πc. Thus $x(M)$ is contained in an N-ball of radius less than πc.

Let $B = B(p, R)$ be an N-ball of smallest radius containing $x(M)$; in particular, $R < \pi c$. By Lemma 1, it follows that $x(M)$ supports B. Let $d = 2c$. Then $R < \pi c = \frac{1}{2}\pi d$.

Let S^n be the n-sphere of curvature $1/d^2$. Let $q \in S^n$ and let $I: N_p \to S^n_q$ be an inner product preserving map. Define the map $\varphi: B \to S^n$ by $\varphi(z) =$

$\exp_q \circ I \circ (\exp_p)^{-1}(z)$. This map takes a metric sphere of radius $r \le R$ at p to a metric sphere of radius r at q, and an angle with measure θ at p to an angle of measure θ at q. Since $x(M)$ supports B, $\varphi \circ x(M)$ supports $\varphi(B)$. By Lemma 3, there are three points, q_1, q_2, q_3, in $\varphi \circ x(M) \cap \partial\varphi(B)$ such that the geodesic triangle in $\varphi(B)$ with q_1, q_2, q_3 as vertices has length L, where

$$L \ge \min\{4R,\, 6d \cdot \sin^{-1}((1/\sqrt{2})\sin(R/d))\}.$$

Let p_1, p_2, p_3 be points in M such that $\varphi \circ x(p_i) = q_i$. Let $L_M(p_1p_2p_3)$ be the length of a geodesic triangle in M with vertices p_1, p_2, p_3; similarly, L_S will refer to the image triangle in S. Then

$$L_M(p_1p_2p_3) \ge L_S(q_1q_2q_3) \ge \min\{4R,\, 6d \cdot \sin^{-1}((1/\sqrt{2})\sin(R/d))\}.$$

The first inequality follows from Proposition 2.

Now, Toponogov's theorem gives $2\pi c \ge L_M(p_1p_2p_3)$. Thus if $R \le \frac{1}{4}\pi d$, we have

$$4R = \min\{4R,\, 6d \cdot \sin^{-1}((1/\sqrt{2})\sin(R/d))\} < 2\pi c$$

or $R < \frac{1}{2}\pi c$. Note that if $\frac{1}{4}\pi d < R \le \frac{1}{2}\pi d$, we have

$$\sin(\pi/6) < (1/\sqrt{2})\sin(R/d)$$

and thus

$$\pi d < 6d \cdot \sin^{-1}((1/\sqrt{2})\sin(R/d)).$$

However, $\frac{1}{4}\pi d < R \le \frac{1}{2}\pi d$ also gives

$$6d \cdot \sin^{-1}((1/\sqrt{2})\sin(R/d)) \le 2\pi c = \pi d,$$

which is a contradiction. Thus $R \le \frac{1}{4}\pi d$ holds and hence also $R \le \frac{1}{2}\pi c$.

Indeed, $R < \frac{1}{2}\pi c$, since $R = \frac{1}{2}\pi c$ could not occur when the minimum of $4R$ is achieved; that is, if $\theta = \pi$ in the proof of Lemma 3. This would mean that two points of $x(M)$ were antipodal on $\partial B(p, R)$, and hence could be joined by an N-geodesic of length πc. But we know that any two points of $x(M)$ can be joined by an N-geodesic of length less than πc. Thus, $R = \frac{1}{2}\pi c$ would contradict $i(N) \ge \pi c$. This completes the proof of Theorem 1.

3. Imbedding and convexity

3.1. Background material. A Jacobi field J along a geodesic $\gamma(s)$ is called *normal* if J is everywhere normal to γ. Let D/ds represent the covariant derivative in N along a given curve. The second fundamental form S of a hypersurface M with respect to a given unit normal field Z is given by $S_p(V, W) = \langle (D/ds(Z))(0), W\rangle$, where V and W are vectors tangent to M at p and γ is a curve in M with $\gamma'(0) = Z_p$.

Suppose γ is normal to a hypersurface M at $\gamma(0)$. Then a Jacobi field J along γ is called an *M-Jacobi field* if $J(0)$ is tangent to M and

$$S_{\gamma(0)}(J(0), J(0)) - DJ|ds(0)$$

is normal to M at $\gamma(0)$, where S is the second fundamental form of the surface M corresponding to the normal $\gamma'(0)$ [16, p. 350]. A *focal point* of M is a point of γ at which a nontrivial M-Jacobi field along γ vanishes.

In the proof of Theorem 2 we shall form a family of immersions by moving along the geodesics normal to a given immersion x. Using properties of this family of immersions to contradict the assumption that x has self-intersections, we conclude that x is an imbedding. We need the following result of Warner on focal points [16, p. 351].

Proposition 3. *Let M be a hypersurface on N, and let γ be a geodesic normal to M and parametrized by arclength. Let a and b, $b > 0$, be real numbers and let t_0 be the smallest positive solution to*

$$\cot(b^{1/2}t) = -a/b^{1/2}.$$

Let $S_{\gamma(0)}$ be the second fundamental form of M with respect to $\gamma'(0)$. Assume all the eigenvalues of $S_{\gamma(0)}$ are greater than or equal to a and all the sectional curvatures of sectons of N which are tangent to γ are less than or equal to b. Then there are no focal points of M on $\gamma|[0, t_0)$.

Our proof also makes use of Jacobi field estimates which are an immediate consequence of Corollary 2A in Karcher's paper [9]. We have the following lemma.

Lemma 4. *Let N be a complete Riemannian manifold with $K(N) \leq 1/d^2$ $(d < \infty)$. Let $J(s)$ be a normal Jaocbi field along a geodesic $\gamma(s)$, with $|J|(0) = 1$ and $|J|'(0) = \lambda > 0$. Then*

$$|J|(S) \geq \cos(s/d) + d\lambda \sin(s/d),$$
$$|J|'(s) \geq \lambda \cos(s/d) - (1/d)\sin(s/d)$$

for $0 \leq s \leq \frac{1}{2}\pi d$.

3.2. Proof of Theorem 2. The main point of this proof is to show that the immersion x is in fact an imbedding. By Theorem 1, $x(M)$ lies strictly inside a ball $B = B(p, R)$ with $R < \frac{1}{2}\pi c$. Choose ε_1 so that $R < \frac{1}{2}(\pi c - \varepsilon_1)$, and d in $(0, \infty)$ so that $1/d^2 \leq 1/(4c^2)$ and $K(N)$ does not exceed $1/d^2$ on $B(p, (\frac{3}{2})\pi c)$. There is a continuous unit normal field Z along the map x (since $x(M)$ lies in an open normal coordinate ball). Since $1/c^2 - 1/d^2 > 0$, Z may be chosen so that the second fundamental form S of x with respect to Z is positive definite. Let

$$\gamma(m, s) = \exp_{x(m)} sZ(m).$$

Thus for each fixed m in M, there is a normal geodesic γ_m at $x(m) = \gamma(m, 0)$ given by $\gamma_m(s) = \gamma(m, s)$. It follows from Proposition 3, with $a = 0$ and $b = 1/d^2$, that if $0 \leq s < \frac{1}{2}\pi d$ and $\gamma_m|[0, s]$ lies in $B(p, (\frac{3}{2})\pi c)$, then there is no focal point of M along $\gamma_m|[0, s]$. Since $2c \leq d$, there is no focal point if $0 \leq s \leq \pi c - \varepsilon_1$.

Since B lies in the open normal coordinate ball of radius $\pi c-\varepsilon_1$ at $x(M)$ for each m in M, there is a first point $g(m)$ where γ_m strikes ∂B. Furthermore, $g(m)$ is not a focal point of M along γ_m.

Since $R < \frac{1}{2}\min\{\pi d, i(N)\}$, it can be shown that $B(p, R)$ is strongly convex. This statement is a consequence of the proof of Theorem 5.14 in Cheeger and Ebin [3, p. 103]. It follows that γ_m cannot be tangent to ∂B at the point $g(m)$; the argument is the same as that given in Proposition 1. Thus γ_m strikes ∂B transversely, and the map $g: M \to \partial B$ is a local diffeomorphism. Since ∂B is diffeomorphic to an $(n-1)$-sphere the map g is a diffeomorphism and hence M is diffeomorphic to an $(n-1)$-sphere.

By the compactness of M there exists an ε_2 such that there are no focal points of M along any normal geodesic for $-\varepsilon_2 < s < \pi c - \varepsilon_1 + \varepsilon_2$. Thus for each fixed s with $-\varepsilon_2 < s < \pi c - \varepsilon_1 + \varepsilon_2$, the mapping $x_s: M \to N$ given by $x_s(m) = \gamma(m, s)$ is an immersion.

Note that, along γ_m, each M-Jacobi field with respect to the immersion x is also an M-Jacobi field with respect to the immersion x_s. We shall write

$$S(V(s), V(s)) = \tfrac{1}{2}D|ds\langle V(s), V(s)\rangle,$$

where V is any such M-Jacobi field along γ_m. For each s, the second fundamental form S_s of x_s with respect to $\gamma_m'(s)$ agrees with S in the sense that

$$S_s(V(s), V(s)) = S(V(s), V(s)).$$

Suppose x is not an imbedding of M into N. Then $x(q_1) = x(q_2)$ for distinct q_1 and q_2 in M. Since $R < \frac{1}{2}\pi c-\varepsilon_1$, the interior of $B(x(m), \pi c-\varepsilon_1)$ contains $x(M)$. Therefore, $\gamma(m, \pi c-\varepsilon_1) \notin x(M)$ for all m in M. Thus there exists t, $0 \le t < \pi c-\varepsilon_1$, such that $x_t(M)\cap x(M) \neq \varnothing$ and $x_s(M)\cap x(M) = \varnothing$ for $t < s \le \pi c - \varepsilon_1$. Hence, there exist distinct points y and z such that $x(z) = x_t(y)$. Furthermore, by the maximality of t, x and x_t are tangent at these two points. Thus we have

$$Z_t(y) = \pm Z(z),$$

where $Z_t(y)$ is $\gamma_y'(t)$, a normal to x_t at $\gamma(y, t)$. The choice $Z_t(y) = Z(z)$ contradicts the fact that g is a diffeomorphism, so $Z_t(y) = -Z(z)$. It follows that both $x(y) = x_t(z)$ and $x(z) = x_t(y)$ hold (see Figure 1).

Let λ_1 be the maximum eigenvalue of the second fundamental form S_0 of x at y and λ_2 be the maximum eigenvalue of S_0 at z. Both λ_1 and λ_2 are positive since S_0 is positive definite everywhere. Set $H = 1/c^2 - 1/d^2$. Then the minimum eigenvalue at y is greater than or equal to H/λ_1 and the minimum eigenvalue at z is greater than or equal to H/λ_2. Note that since $1/(4c^2) \le 1/d^2$ we have

$$H \ge 3/d^2. \tag{1}$$

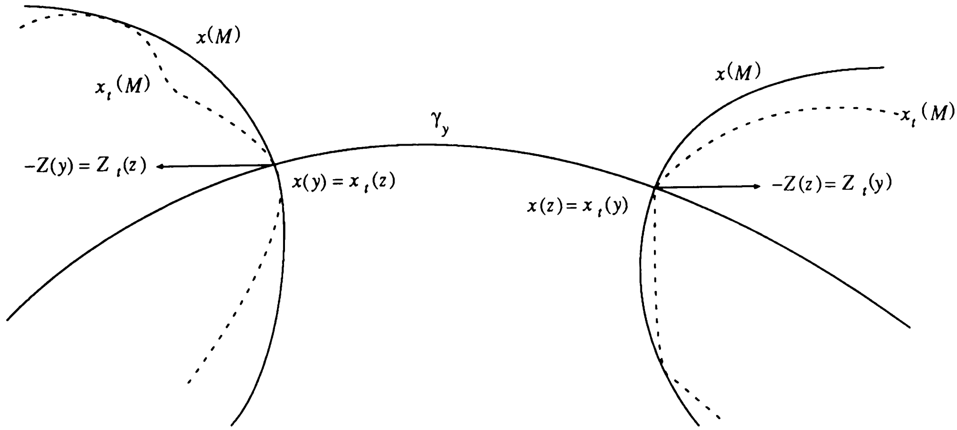

FIGURE 1

By the minimality of t, the image of x_t near z must lie on the side of $x(M)$ determined by $Z_t(y) = -Z(z)$, as Figure 1 indicates.

Thus,

$$(1/|V_i^2|)S(V_i, V_i) \leq -H/\lambda_i, \qquad i = 1, 2,$$

for all V_1 tangent to x_t at y and for all V_2 tangent to x_t at z. In particular, let $V_i(s)$ be the M-Jacobi field along the geodesic γ_t given by

$$V_i(s) = (x_s)_* V_i, \qquad i = 1, 2,$$

where V_1 is a unit vector corresponding to the maximal eigenvalue, λ_1, of S_0 at y and V_2 is a unit vector corresponding to the maximal eigenvalue, λ_2, of S_0 at z. Then we have, for $i, j = 1, 2$ and $i \neq j$,

$$(1/|V_i(t)|)D|ds(|V_i(t)|) = (1/|V_i(t)|^2)S(V_i(t), V_i(t)) \leq -H/\lambda_j,$$

By Lemma 4, for $i, j = 1, 2$ and $i \neq j$, we have

$$(1/|V_i|(t))(\lambda_i \cos(t/d) - (1/d)\sin(t/d)) \leq -H/\lambda_j.$$

Since $t < \frac{1}{2}\pi d$, we conclude that

$$\sin(t/d)/(|V_i(t)|d) > H/\lambda_j. \tag{2}$$

Also by Lemma 4, for $i = 1, 2$,

$$|V_i(t)| \geq d\lambda_i \sin(t/d) + \cos(t/d) > d\lambda_i \sin(t/d). \tag{3}$$

Combining formulas (2) and (3) gives $1/(d^2\lambda_1) > H/\lambda_2$ and $1/(d^2\lambda_2) > H/\lambda_1$.

Thus by (1), we have $\lambda_1 \geq 3\lambda_2$ and $\lambda_2 \geq 3\lambda_1$, which is impossible since λ_1 and λ_2 are positive. Hence x must be an imbedding.

Since M is diffeomorphic to an $(n-1)$-sphere and is imbedded by x in the interior of $B(p, R)$, then $x(M)$ bounds a well-defined interior, $\operatorname{Int} x(M)$, by the generalized Jordan-Brouwer Separation Theorem [11, vol. 1, p. 591].

Furthermore, $B(p, R)$ is strongly convex. Hence we need only to show that the unique minimal geodesic joining any two points of $\operatorname{Int} x(M)$ lies in $\operatorname{Int} x(M)$.

The following proof, which is included for completeness, is well known and is attributed to Erhard Schmidt [8, p. 241]. Let y and z be two points in $\operatorname{Int} x(M)$ and let γ be the geodesic segment joining them. Suppose γ intersects $x(M)$. Since $\operatorname{Int} x(M)$ is connected, there is a path $\alpha(t)$, $0 \le t \le a$, from y to z entirely in $\operatorname{Int} x(M)$. Let β_t be the geodesic segment from y to $\alpha(t)$. There is a smallest t such that β_t intersects $x(M)$, say $t = b$. Let q be the first point of β_b which lies in $x(M)$. Since b is the smallest possible, β_b must lie in $x(M) \cup \operatorname{Int} x(M)$ and β_b must be tangent to $x(M)$ at q. However, since S is positive definite at q, any geodesic segment which is tangent to $x(M)$ at q locally lies outside of $x(M)$. This contradiction shows that γ must lie in $\operatorname{Int} x(M)$, and hence that $x(M)$ bounds a convex body. This completes the proof of Theorem 2.

4. Tightness of bounds

Finally we shall verify that the bound $R < \frac{1}{2}\pi c$, given in Theorem 1, is the best possible. Let N be a simply connected space of constant curvature less than or equal to $1/(4c^2)$. For each $\varepsilon > 0$, we construct a surface M, isometrically imbedded into N, such that $K(M) \ge 1/c^2$ and M lies in no N-balls of radius $\pi c - \varepsilon$.

If N is a Euclidean n-space it is well known that there is a family of incomplete surfaces of revolutions with curvature $1/c^2$ whose extrinsic diameters approach πc. By rounding the corners, one achieves a family of surfaces with curvature at least $1/c^2$.

If N has constant curvature $1/d^2$, with $1/d^2 \le 1/(4c^2)$, or constant curvature $-1/d^2$, then a family of surfaces of rotation can be generated in a similar manner. In a 2-dimensional totally geodesic submanifold of N, let γ be a geodesic. Choose a point p on γ and a parallel unit normal field V along γ. An orthogonal grid is formed by the geodesics tangent to V and the curves γ_r formed by the points of distance r along these geodesics. Let g represent the distance along γ and r the distance perpendicular to γ.

For $0 < a \le c$ and $-\frac{1}{2}\pi c \le s \le \frac{1}{2}\pi c$, choose $r_a(s)$ and $g_a(s)$ to satisfy the equations

$$d \cdot \sin(r_a(s)/d) = a \cdot \cos(s/c),$$

$$g_a(s) = \int_0^s (1/\cos(r_a/d))(1 - (dr_a/dt)^2)^{1/2}\, dt$$

if $K(N) = 1/d^2$; or the equations

$$d \sinh(r_a(s)/d) = a \cdot \cos(s/c),$$

$$g_a(s) = \int_0^s (1/\cosh(r_a/d))(1 - (dr_a/dt)^2)^{1/2}\, dt$$

if $K(N) = -1/d^2$. Let α_a be the curve given in the (g, r)-coordinate system by $\alpha_a(s) = (g(s), r_a(s))$. Calculations show that the integrals are well defined, $g_a(s)$ is a regular increasing function of s, and that s is the arclength parameter of α_a.

The map $s \to \alpha_a(s)$ is regular and one-to-one since there are no focal points of γ along any geodesic normal to within distance $r(s)$ of γ. Thus a surface of rotation, M_a, is generated in N when α_a is rotated about γ in some 3-dimensional totally geodesic submanifold. Calculations show that these surfaces have constant curvature $1/c^2$, except at the endpoints of α_a. The surfaces of rotation M_a approach γ as a approaches 0. Thus, by rounding the surfaces at the endpoints of α_a we have a family of surfaces whose extrinsic diameters approach πc and whose sectional curvatures are at least $1/c^2$, as desired.

Note that the restriction $i(N) \geq \pi c$ is necessary in Theorem 2. The conclusion there is that an isometrically imbedded hypersurface M, with $K(M) \geq 1/c^2$, bounds a convex body of N. Consider our example in R^n with the surfaces rotated about the x-axis, and identify x with $x + 2k(\pi c - \varepsilon)$, k an integer. The resulting flat manifold has injectivity radius $\pi c - \varepsilon$. A surface constructed as above whose extrinsic diameter exceeds $\pi c - \varepsilon$ will not bound a convex body in the flat manifold since the shortest geodesic joining a pair of points near the ends of the diameter lies outside the body which the surface bounds.

As has already been mentioned, do Carmo and Warner showed that an isometric immersion x of an $(n-1)$-manifold M, with $K(M) \geq 1/c^2$, into a sphere of curvature $1/c^2$ is an imbedding and that either $x(M)$ is totally geodesic or it bounds a convex body in N. So the question still remains whether or not our restriction $K(N) \leq 1/(4c^2)$ can be improved in general. Furthermore, if $x(M)$ is not required to bound a convex body, one can ask whether $i(N) \geq \pi c$ can be replaced by $i(N) > \frac{1}{2}\pi c$ or some other restriction.

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